A SEMINAR ON
" AN ANALYSIS: ROOTS OF EQUATION BY OPEN METHODS"

Presented by: Dr Arun Kumar Tripathy

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DEPARTMENT OF MATHEMATICS, PATTAMUNDAI COLLEGE, PATTAMUNDAI

REPORT

A seminar was organised by Department of Mathematics ,Pattamundai College, Pattamundai on 23.02.2019 on the topic " AN ANALYSIS: ROOTS OF EQUATION BY OPEN METHODS".Dr Arun Kumar Tripathy , Lecturer in Mathematics, S. S. B. College, Mahakalpara was resource person for the seminar.In this seminar most of the students were present .Prof Ramesh Chandra Sahoo , Principal, chaired the meeting. Prof Arabinda Pandab, Head of the Department gave a key note address of the topic and welcomed the guests on the diace and the participants. The meeting was ended with vote of thanks by Dr Nirmal Kumar Sahoo, another faculty member.

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AN ANALYSIS: ROOTS OF EQUATION BY OPEN METHODS

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1.1 Introduction

In this lecture, we will discuss numerical methods for the **Root-Finding Problem**. As the title suggests, the Root-Finding Problem is the problem of finding a root of the equation $f(x) = 0$, where $f(x)$ is a function of a single variable x. Specifically, the problem is stated as follows:

The Root-Finding Problem

Given a function $f(x)$, find $x = \xi$ such that $f(\xi) = 0$.

The number ξ is called a **root** of $f(x) = 0$ or a **zero** of the function $f(x)$. The function of $f(x)$ may be algebraic or trigonometric functions. **The well-known examples of algebraic functions are polynomials.** A polynomial of degree n is written as $P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. The numbers $a_i, i = 0, \cdots, n$ are called the coefficients of the polynomial. If $a_n = 1$, the polynomial $P_n(x)$ is called the **monic polynomial.**

Applications of the Root-Finding Problem
The root-finding problem is one of the most important computational problems. It arises in a wide variety of practical applications in **physics, chemistry, biosciences, engineering**, etc. As a matter of fact, determination of any unknown appearing implicitly in scientific or engineering formulas, gives rise to a root-finding problem. We consider one such simple application here.

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1.2 Bisection-Method

As the title suggests, the method is based on repeated bisections of an interval containing the root. The basic idea is very simple.

Basic-Idea:
Suppose $f(x) = 0$ is known to have a real root $x = \xi$ in an interval $[a, b]$.

- Then **bisect** the interval $[a, b]$, and let $c = \dfrac{a+b}{2}$ be the middle point of $[a, b]$. If c is the root, then we are done. Otherwise, one of the intervals $[a, c]$ or $[c, b]$ will contain the root.

- Find the one that contains the root and bisect that interval again.

- Continue the process of bisections until the root is trapped in an interval as small warranted by the desired accuracy.

To implement the idea, we need to know which of the two intervals in each iteration contains the root of $f(x) = 0$. The **Intermediate Mean-Value Theorem** of calculus can help us identify the interval in each iteration. For a proof of this theorem, see any calculus book.

Intermediate Mean-Value Theorem (IMV Theorem)

Let $f(x)$ be a continuous function defined on $[a, b]$, such that

$f(a)$ and $f(b)$ are of opposite signs (i.e., $f(a)f(b) < 0$).

Then there is a root $x = c$ of $f(x) = 0$ in $[a, b]$.

Algorithm 1.1 The Bisection Method for Rooting-Finding

Inputs: (i) $f(x)$ - The given function

(ii) a_0, b_0 - The two numbers such that $f(a_0)f(b_0) < 0$.

Output: An approximation of the root of $f(x) = 0$ in $[a_0, b_0]$.
For $k = 0, 1, 2, \ldots$, do until satisfied:

- Compute $c_k = \dfrac{a_k + b_k}{2}$.
- Test if c_k is the desired root, if so, stop.
- If c_k is not the desired root, test if $f(c_k)f(a_k) < 0$. If so, set $b_{k+1} = c_k$ and $a_{k+1} = a_k$. Otherwise, set $a_{k+1} = c_k$, $b_{k+1} = b_k$.

End.

Example 1.1 $f(x) = x^3 - 6x^2 + 11x - 6$.
Let $a_0 = 2.5$, $b_0 = 4$. *Then* $f(a_0)f(b_0) < 0$. *Then there is a root in* $[2.5, 4]$.

Iteration 1. $k = 0$:

$c_0 = \dfrac{a_0 + b_0}{2} = \dfrac{4 + 2.5}{2} = \dfrac{6.5}{2} = 3.25.$
Since $f(c_0)f(a_0) = f(3.25)f(2.5) < 0$, set $b_1 = c_0,\ a_1 = a_0$.

Iteration 2. $k = 1$:
$c_1 = \dfrac{3.25 + 2.5}{2} = 2.8750.$
Since $f(c_1)f(a_1) > 0$, set $a_2 = 2.875,\ b_2 = b_1$

Iteration 3. $k = 2$:
$c_2 = \dfrac{a_2 + b_2}{2} = \dfrac{2.875 + 3.250}{2} = 3.0625$

Since $f(c_2)f(a_2) = f(3.0625)f(2.875) < 0$, set $b_3 = c_2,\ a_3 = a_2$.

Iteration 4. $k = 3$
$c_3 = \dfrac{a_3 + b_3}{2} = \dfrac{2.875 + 3.0625}{2} = 2.9688.$
It is clear that the iterations are converging towards the root $x = 3$.
Notes:

1. From the statement of the Bisection algorithm, it is clear that **the algorithm always converges.**

2. The example above shows that the convergence, however, can be very slow.

Stopping Criteria
Since this is an iterative method, we must determine some stopping criteria that will allow the iteration to stop. Here are some commonly used stopping criteria.
Let ϵ be the tolerance; that is, we would like to obtain the root with an error of at most of ϵ. Then

Accept $x = c_k$ as a root of $f(x) = 0$ if any of the following criteria is satisfied:

1. $|f(c_k)| \leq \epsilon$ (*The functional value is less than or equal to the tolerance*).

2. $\dfrac{|c_{k-1} - c_k|}{|c_k|} \leq \epsilon$ (*The relative change is less than or equal to the tolerance*).

3. $\dfrac{(b-a)}{2^k} \leq \epsilon$ (*The length of the interval after k iterations is less than or equal to the tolerance*).

4. *The number of iterations k is greater than or equal to a predetermined number, say N.*

Comments:

1. Criterion 1 can be misleading, since, it is possible to have $|f(c_k)|$ very small, even if c_k is not close to the root. **(Do an example to convince yourself that it is true).**

2. Criterion 3 is based on the fact that after k steps, the root will be computed with error at most $\left(\frac{b_0 - a_0}{2^k}\right)$.

Number of Iterations Needed in the Bisection Method to Achieve Certain Accuracy

Let's now find out what is the minimum number of iterations N needed with the Bisection method to achieve a certain desired accuracy.

Criterion 3 can be used to answer this. The interval length after N iterations is $\frac{b_0 - a_0}{2^N}$.

So, to obtain an accuracy of ϵ, using the Criterion 3, we must have $\frac{b_0 - a_0}{2^N} \leq \epsilon$.

$$\text{That is,} \quad 2^{-N}(b_0 - a_0) \leq \epsilon$$

$$\text{or} \quad 2^{-N} \leq \frac{\epsilon}{(b_0 - a_0)}$$

$$\text{or} \quad -N \log_{10} 2 \leq \log_{10}\left(\frac{\epsilon}{b_0 - a_0}\right)$$

$$\text{or} \quad N \log_{10} 2 \geq -\log_{10}\left(\frac{\epsilon}{b_0 - a_0}\right).$$

$$\text{or} \quad N \geq \frac{-\log_{10}\left(\frac{\epsilon}{b_0 - a_0}\right)}{\log_{10} 2}.$$

$$\text{or} \quad N \geq \frac{[\log_{10}(b_0 - a_0) - \log_{10}(\epsilon)]}{\log_{10} 2}$$

Theorem 1.1 *The number of iterations N needed in the bisection method to obtain an accuracy of ϵ is given by*

$$N \geq \frac{[\log_{10}(b_0 - a_0) - \log_{10}(\epsilon)]}{\log_{10} 2}$$

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Example 1.2 *Suppose we would like to determine a priori the minimum number of iterations needed in the bisection algorithm, given $a_0 = 2.5, b_0 = 4$, and $\epsilon = 10^{-3}$. By Theorem 1.1, we have*

$$N \geq \frac{\log_{10}(1.5) - \log_{10}(10^{-3})}{\log_{10} 2} = \frac{\log_{10}(1.5) + 3}{\log_{10}(2)} = 10.5507.$$

Thus, a minimum of 11 iterations will be needed to obtain the desired accuracy using the Bisection Method.

Remarks: Since the number of iterations N needed to achieve a certain accuracy depends upon the initial length of the interval containing the root, it is desirable to choose the initial interval $[a_0,\ b_0]$ as small as possible.

1.3 Fixed-Point Iteration

A number ξ is a **fixed point** of a function $g(x)$ if $g(\xi) = \xi$.

Suppose that the equation $f(x) = 0$ is written in the form $x = g(x)$; that is, $f(x) = x - g(x) = 0$. *Then any fixed point ξ of $g(x)$ is a root of $f(x) = 0$*; because $f(\xi) = \xi - g(\xi) = \xi - \xi = 0$. *Thus a root of $f(x) = 0$ can be found by finding a fixed point of $x = g(x)$, which corresponds to $f(x) = 0$.*

Finding a root of $f(x) = 0$ by finding a fixed point of $x = g(x)$ immediately suggests an iterative procedure of the following type.

Start with an initial guess x_0 of the root, and form a sequence $\{x_k\}$ defined by

$$x_{k+1} = g(x_k),\ k = 0, 1, 2, \cdots$$

If the sequence $\{x_k\}$ converges, then $\lim_{k\to\infty} x_k = \xi$ will be a root of $f(x) = 0$.

The question therefore rises:

Given $f(x) = 0$, how to write $f(x) = 0$ in the form $x = g(x)$, so that starting with any x_0 in $[a,\ b]$, the sequence $\{x_k\}$ defined by $x_{k+1} = g(x_k)$ is guaranteed to converge?

The simplest way to write $f(x) = 0$ in the form $x = g(x)$ is to add x on both sides, that is,

$$x = f(x) + x = g(x).$$

But it does not very often work.

To convince yourself, consider **Example 1.1** again. Here

$$f(x) = x^3 - 6x^2 + 11x - 6 = 0.$$

Define $g(x) = x + f(x) = x^3 - 6x^2 + 12x - 6$. We know that there is a root of $f(x)$ in $[2.5,\ 4]$; namely $x = 3$. Let's start the iteration $x_{k+1} = g(x_k)$ with $x_0 = 3.5$,

then we have:

$$\begin{aligned}
x_1 &= g(x_0) = g(3.5) = 5.3750\\
x_2 &= g(x_1) = g(5.3750) = 40.4434\\
x_3 &= g(x_2) = g(40.4434) = 5.6817 \times 10^4\\
x_4 &= g(x_3) = g(5.6817 \times 10^4) = 1.8340 \times 10^{14}
\end{aligned}$$

The sequence $\{x_k\}$ is clearly diverging.

The convergence and divergence of the fixed-point iteration are illustrated by the following graphs.

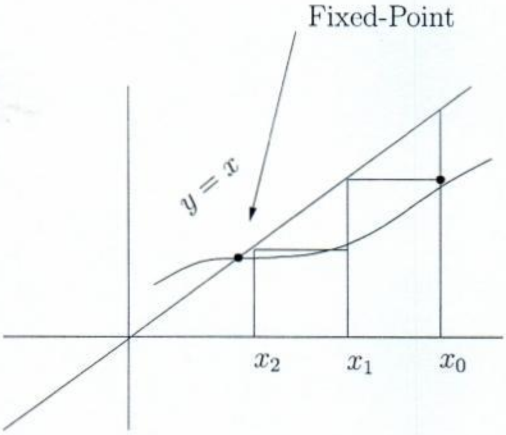

Figure 1.1: Convergence of the Fixed-Point Iteration

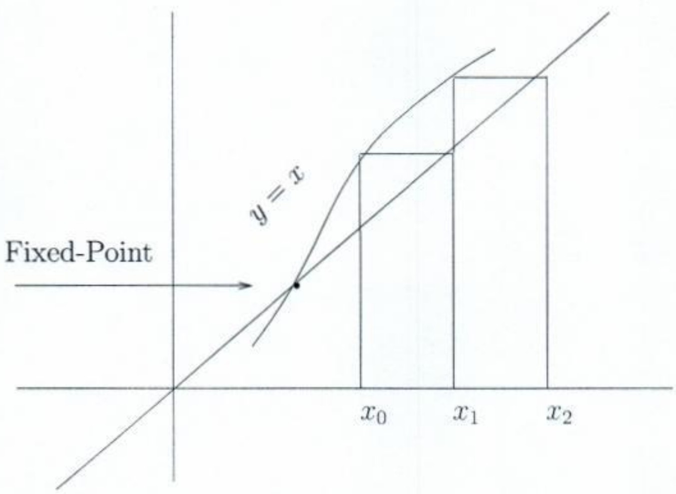

Figure 1.2: Divergence of the Fixed-Point Iteration

The following theorem gives a sufficient condition on $g(x)$ which ensures the convergence of the sequence $\{x_k\}$.

Theorem 1.2 (Fixed-Point Iteration Theorem): *Let $f(x) = 0$ be written in the form $x = g(x)$. Assume that $g(x)$ has the following properties:*

1. *For all x in $[a,\ b]$, $g(x) \in [a,\ b]$; that is $g(x)$ takes every value between a and b.*

2. *$g'(x)$ exists on $(a,\ b)$ with the property that there exists a positive constant $r < 1$ such that*

$$|g'(x)| \leq r,$$

for all x in (a, b).

Then

(i) there is a unique fixed point $x = \xi$ of $g(x)$ in $[a, b]$.

(ii) For any x_0 in $[a,\ b]$, the sequence $\{x_k\}$ defined by

$$x_{k+1} = g(x_k),\ k = 0, 1, \cdots$$

converges to the fixed point $x = \xi$; that is to the root ξ of $f(x) = 0$.

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The proof of the Theorem requires the **Mean Value Theorem** of Calculus;

The Mean Value Theorem (MVT)

Let $f(x)$ be a continuous function on $[a,\ b]$, and be differentiable on (a, b).

Then there is a number c in (a, b) such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Proof of the Fixed-Point Theorem:
The proof comes in three parts: **Existence, Uniqueness, and Convergence**. We first prove that there is a root of $f(x) = 0$ in $[a,\ b]$ and it is unique. Since a fixed point of $x = g(x)$ is a root of $f(x) = 0$, this amounts to proving that there is a fixed point in $[a,\ b]$ and it is unique. We then prove that the sequence $x_{k+1} = g(x_k)$ converges to the root.

Existence:
If $a = g(a)$, then a is a fixed point. If $b = g(b)$, then b is a fixed point. If not, **because of the assumption** 1, $g(a) > a$, and $g(b) < b$. Thus, $f(a) = g(a) - a > 0$, and $f(b) = g(b) - b < 0$. Also $f(x) = g(x) - x$ is continuous on $[a, b]$.
Thus, by applying the **Intermediate Mean Value Theorem** to $f(x)$ in $[a, b]$, we conclude that there is a root of $f(x)$ in $[a,\ b]$. **This proves the existence.**
Uniqueness (by contradiction):
To prove uniqueness, suppose ξ_1 and ξ_2 are for two fixed points in $[a,\ b]$, and $\xi_1 \neq \xi_2$.
Now, apply the MVT to $g(x)$ in $[\xi_1, \xi_2]$, we can find a number c in (ξ_1, ξ_2) such that

$$
\begin{aligned}
x_0 &= 0 \\
x_1 &= \cos x_o = 1 \\
x_2 &= \cos x_1 = 0.54 \\
x_3 &= \cos x_2 = 0.86 \\
&\vdots \\
x_{17} &= 0.73956 \\
x_{18} &= \cos x_{17} = 0.73955
\end{aligned}
$$

1.4 The Newton-Raphson Method

The Newton-Raphson method, described below, shows that there is a special choice of $g(x)$ that will allow us to readily use the results of Corollary 1.1.
Assume that $f''(x)$ exists and is continuous on $[a,\ b]$ and ξ is a **simple root** of $f(x)$, that is, $f(\xi) = 0$ and $f'(\xi) \neq 0$.

Choose

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Then $\quad g'(x) = 1 - \dfrac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \dfrac{f(x)f''(x)}{(f'(x))^2}.$

Thus, $g'(\xi) = \dfrac{f(\xi)f''(\xi)}{(f'(\xi))^2} = 0$, since $f(\xi) = 0$, and $f'(\xi) \neq 0$.

Since $g'(x)$ is continuous, this means that there exists a small neighborhood around the root $x = \xi$ such that for all points x in that neighborhood, $|g'(x)| < 1$.

Thus if $g(x)$ is chosen as above and the **starting approximation** x_0 **is chosen sufficiently close to the root** $x = \xi$, then the fixed-point iteration is guaranteed to converge, according to the corollary above.

This leads to the following well-known classical method, known as, the **Newton-Raphson** method.

(**Remark:** In many text books and literature, the Newton-Raphson Method is simply called the **Newton Method**. But, the title "The Newton-Raphson Method" is justified. See an article by [ ] in SIAM Review (1996)).

Algorithm 1.2 ***The Newton-Raphson Method***

Inputs: $f(x)$ - The given function
x_0 - The initial approximation
ϵ - The error tolerance
N - The maximum number of iterations

Output: An approximation to the root $x = \xi$ or a message of failure.

Assumption: $x = \xi$ is a simple root of $f(x)$.

For $k = 0, 1, \cdots$ do until convergence or failure.

- Compute $f(x_k), f'(x_k)$.
- Compute $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$.
- Test for convergence or failure:

$$\left.\begin{array}{ll} \text{If} & |f(x_k)| < \epsilon \\ \text{or} & \dfrac{|x_{k+1} - x_k|}{|x_k|} < \epsilon \end{array}\right] \text{stopping criteria.}$$

or $k > N$, Stop.

End

Definition 1.1 *The iterations* $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$ *are called* ***Newton's iterations****.*

Remarks:

1. If none of the above criteria has been satisfied within a predetermined, say N, iterations then the method has failed after the prescribed number of iterations. In that case, one could try the method again with a different x_0.
2. A judicious choice of x_0 can sometimes be obtained by drawing the graph of $f(x)$, if possible. However, there does not seem to exist a clear-cut guideline of how to choose a right starting point x_0 that guarantees the convergence of the Newton-Raphson Method to a desired root.

Some Familiar Computations Using the Newton-Raphson Method

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Students Attendance on the seminar"An Analysis : Roots of Equation by Open methods " on 23.02.2019

Sl No	Roll No	Signature of the Student
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4	BS-16-159	suchitra sahoo
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6	BS-16-136	Harapriya Ghadei
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8	BS-18-080	Usharani Mohanty
9	BS-18-074	Anuradha Rout
10	BS-18-064	Sreehree Mahanta.
11	BS-18-026	Amrita Bhuyan.
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13	BS-18-128	sarwotika Rath.
14	BS-18-004	Anyasha Panda
15	BS-18-139	Manisha Swain
16	BS-18-138	Elina Swain
17	BS-17-104	Swagatika Das.
18	BS-17-054	sagarina Nayak
19	BS-17-019	Swagatika Nayak
20	BS-17-119	Manasi Mohanty
21	BS-18-070	Malaya RanJan swain.
22	BS-18-107	Satyajit Mallick
23	BS-17-158	Debabrata Khadua
24	BS-17-029	Nandakishore Rout.
25	BS-17-098	SK. Asafullah
26	BS-17-~~030~~ 132	Dibyasundar Behera
27	BS-17-135	Ansuman Sahoo
28	BS-18-075	Kailash chandra Mallik
29	BS-18-091	Satya Narayan Das
30	BS-18-065	Subhrajit Rout
31	BS-18-105	Yamunakanta Rout
32	BS-16-096	Chandra sekhar Rout
33	BS-16-123	BISWAJIT pradhan.

34	BS16-051	Soumyaranjan Sahoo.
35	BS16-020	Ashoka Kumar Sethi
36	BS17-068	Swaraj Kumar Das
37	BS17-069	Rajanikanta Jena
38	BS16-046	Chandrabhanu Patra
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